HALVES, PIECES, AND TWOTHS: CONSTRUCTING REPRESENTATIONAL CONTEXTS IN TEACHING FRACTIONS[1]

Deborah Loewenberg Ball[2]

Goals of Teaching and Learning Mathematics

Current discourse about the desirable ends of mathematics teaching and learning centers on the development of mathematical understanding and mathematical power——the capacity to make sense with and about mathematics (cf. California State Department of Education, 1985; National Council of Teachers of Mathematics, 1989a; National Research Council, 1989). Learning mathematics with understanding, according to this view, entails making connections between informal understandings——about mathematical ideas, quantitative and spatial patterns, and relationships——and more formal mathematical ideas. Connections must be forged among mathematical ideas (Fennema, Carpenter, and Peterson, 1989). Students must develop the tools and dispositions to frame and solve problems, reason mathematically, and communicate about mathematics (National Council of Teachers of Mathematics, 1989a).

These goals go beyond understanding of particular ideas——place value, functions, triangles, area measurement. "Knowing mathematics" includes knowing how to *do* mathematics: "To know mathematics is to investigate and express relationships among patterns, to be able to discern patterns in complex and obscure contexts, to understand and transform relationships among patterns" (National Research Council, 1990, p. 12). Included in this view of understanding mathematics also are ways of seeing, interpreting, thinking, doing, and communicating that are special to the community of mathematicians. These specialized skills and ways of framing and solving problems can contribute to everyday confidence and competence; they are personally as well as intellectually empowering. Schoenfeld (1989) summarizes this dimension of mathematical knowledge:

> Learning to think mathematically means (a) developing a mathematical point of view——valuing the process of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of understanding structure——mathematical sense-making. (p. 9)

This sense-making is both individual and consensual, for mathematical knowledge is socially constructed and validated. Drawing mathematically reasonable conclusions involves the capacity to make

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[2]Deborah Loewenberg Ball, assistant professor of teacher education at Michigan State University, is a senior researcher with the National Center for Research on Teacher Education.

mathematically sound arguments to convince oneself and others of the plausibility of a conjecture or solution. It also entails the capacity to appraise and react to others' reasoning and to be willing to change one's mind for good reasons.

An Epistemology of Teaching Mathematics for Understanding

Contemplating Content and Students

Helping students develop this kind of mathematical power depends on insightful consideration of both content and learners, consideration that is both general and situated. Figuring out how to help students develop this kind of mathematical knowledge depends on a careful analysis of the specific content to be learned: the ideas, procedures, and ways of reasoning. Such analyses must examine the particular: Probability, for instance, is a domain that differs in some important ways from number theory, both in the nature of the ideas themselves and in their justification, as well as in the kinds of reasoning entailed. Similarly, an argument in geometry is distinctive from one in arithmetic. Differences in how a given topic evolved may also be useful in considering how students may encounter and develop its ideas: That it took the mathematical community centuries to accept negative numbers in a "felt way" (Kline, 1970) may help to explain students' struggles to make sense of quantities that are less than zero (Ball, 1990b).

But analyzing the content——concepts and ways of knowing——is insufficient. Helping students develop the kind of knowledge described above also depends crucially on understandings of students themselves and how they learn the particular content. Careful analyses of the content cannot suffice to map the terrain through the eyes of the prospective child-explorer. As Dewey (1902) puts it aptly, "The map does not take the place of the actual journey" (p. 20). The teacher must simultaneously maintain a complex and wide-angled view of the territory, all the while trying to see it through the eyes of the learner exploring it for the first time (Lampert, in preparation). How does the mathematics appear to a nine-year-old? Nine-year-olds' ideas and ways of thinking approach formal mathematical ideas and ways of thinking unpredictably and, at times, with breathtaking elegance. Teachers, argues Hawkins (1972), must be able to "sense when a child's interests and proposals . . . are taking him near to mathematically sacred ground" (p. 113). This bifocal perspective——perceiving the mathematics through the mind of the learner while perceiving the mind of the learner through the mathematics——is central to the teacher's role in helping students learn with understanding.

Representational Contexts for Learning Mathematics

But this contemplation of content and students is not passive. The teacher is not, as Hawkins (1972), points out, simply an observer; the teacher's role is to *participate* in students' development:

> As a diagnostician, the teacher is trying to map into his own the momentary state and trajectory of another mind and then, as provisioner, to enhance (not replace) the resources of that mind from his own store of knowledge and skill. (p. 112)

In order to help students develop mathematical understanding and power, the teacher must select and construct models, examples, stories, illustrations, and problems that can foster students' mathematical development. Lampert (1989) writes of the need to select a representational domain with which the children are familiar and in which they are competent to make sense——in other words, in which they can extend and develop their understandings of the ideas, as well as their capacity to reason with and about those ideas. For instance, because students are familiar with relationships among pennies, dimes, and dollars, and because they are comfortable with the notation, Lampert argues that money may provide one helpful terrain in which they can extend their understanding of decimal numeration. Dewey (1902) writes:

> What concerns [the teacher] is the ways in which that subject may become part of experience; what there is in the child's present that is usable in reference to it; how such elements are to be used; how his own knowledge of the subject-matter may assist in interpreting the child's need and doings, and *determine the medium* in which the child should be placed in order that his growth may be properly directed. [The teacher] is concerned, not with the subject-matter as such, but with the subject-matter as a related factor in a total and growing experience. (p. 23, emphasis added)

The issue of selecting, developing, and shaping instructional representations has been the focus of a wide range of inquiry (e.g., Ball, 1988; Kaput, 1987, 1988; Lampert, 1986, 1989; Lesh, Behr, and Post, 1987; Lesh, Post, and Behr, 1987; McDiarmid, Ball, and Anderson, 1989; Wilson, 1988; Wilson, Shulman, and Richert, 1987). Shulman (1986) and his colleagues (Wilson, Shulman, and Richert, 1987) have developed a construct which they call *pedagogical content knowledge*: an "amalgam" of knowledge of subject matter and students, of knowledge and learning. Pedagogical content knowledge includes understandings about what students find interesting and difficult as well as a repertoire of representations, tasks, and ways of engaging students in the content. Nesher (1989) frames the problem for the teacher of mathematics in terms of two main needs: "(a) the need for a young child to construct his knowledge through interaction with the environment, and (b) the need to arrive at mathematical truths" (p. 188). The teacher must structure what Nesher calls a "learning system"——in which learners can explore and test mathematical ideas. Nesher's framework reminds us that the representation of ideas is more than just a catalog of ideas or a series of models——rather it is interactive and takes place within a larger context of ideas, individuals, and their discourse.

Dewey's (1902) problem of "determining the medium," or weaving what I will call a *representational context* in which children can *do*——explore, test, reason, and argue about——and consequently, *learn*, particular mathematical ideas and tools is at the heart of the difficult work of teaching for understanding in mathematics. Such representational contexts must balance respect for the integrity and spirit of mathematics with an equal and serious respect for learners, serving as an "anchor" for the development of learners' mathematical ideas, tools, and ways of reasoning. These contexts must provide rich opportunities for both individual and group discourse. All this sounds both sensible and elegant——pulling it off, however, is difficult.

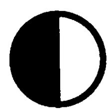
Learning to Teach Mathematics for Understanding

Learning to teach mathematics for understanding is not easy. This paper examines two reasons for this. First, practice itself is complex. Constructing and orchestrating fruitful representational contexts, for example, is inherently difficult and uncertain, requiring considerable knowledge and skill. Second, many teachers' traditional experiences with and orientations to mathematics and its pedagogy hinder their ability to conceive and enact a kind of practice that centers on mathematical understanding and reasoning and that situates skill in context. Helping teachers develop their practice in the direction of teaching mathematics for understanding requires a deep respect for the complexity of such teaching and depends on taking teachers seriously as learners. In this paper I explore and provide evidence for this claim.

Creating and Orchestrating Fruitful Representational Contexts[3]

The deliberations entailed in constructing a viable representational context draw on multiple kinds of knowledge: of the mathematical content, of students and how they learn, of the particular setting.

Considering the content. Substantively, a representation should make prominent conceptual dimensions of the content at hand, not just its surface or procedural characteristics. Important to bear in mind is that representations are metaphorical, borrowing meaning from one domain to clarify or illuminate something in another. As with metaphors——where objects are never isomorphic with their comparative referents——mathematical ideas are by definition broader than any specific representation. For example, area models——such as a circle model of 1/2:



represent only one of several meanings of fractions (Ohlsson, 1988). Despite the fact that this is *the* most frequent representation that children will give if asked what one-half means, 1/2 also refers to the point halfway between 0 and 1 on a number line, the ratio of one day of sunshine to every two of clouds, or the probability of getting one true-false test item right.

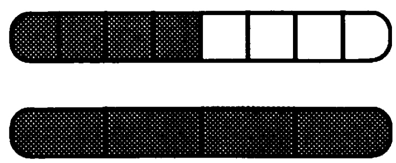
No representational context is perfect. A particular representation may be skewed toward one meaning of a mathematical idea, obscuring other, equally important ones. For example, the number line as a context for exploring negative numbers highlights the positional or absolute value aspect of integers: that -5 and 5 are each five units away from 0. It does not necessarily help students come to grips with the idea that -5 *is less than* 5. Using bundling sticks to explore multidigit addition and subtraction directs attention to the centrality of *grouping* in place value, but may hide the importance of the positional nature of our decimal number system.

Beyond the substance of the topic itself, another layer of complexity rests with the fact that representation is fundamental to mathematics itself (Kaput, 1987; Putnam, Lampert, and Peterson, 1990). One power of mathematics lies in its capacity to represent important relationships and patterns in

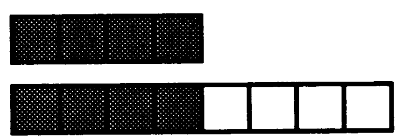
[3] My thinking about representations in teaching has been influenced by conversations over time with Suzanne Wilson and G. Williamson McDiarmid. Wilson's (1988) work on representations in the teaching of U.S. history as well as my work with McDiarmid (cf. McDiarmid, Ball, and Anderson, [1989]) have also extended my ideas about this aspect of teaching for understanding. In addition, the conversations I have had with Sylvia Rundquist over the past two years——in particular the insightful questions she asks me about my teaching——have contributed significantly to my work on representation.

ways that enable the knower to generalize, abstract, analyze, understand. Learning to represent is therefore a goal of mathematics instruction, not just a means to an end. The teacher must figure out ways to help students learn to build their *own* models and representations——of real world phenomena as well as of mathematical ideas (Putnam, Lampert, and Peterson, 1990).

In teaching fractions, the teacher must weigh the relative advantages in providing students with structured representational materials (such as fraction bars that are already ruled into certain fixed partition sets) versus having students refine existing models and develop their own representational media (e.g., drawing circular regions and subdividing portions thereof). Take the idea of *unit*, which is central to fraction knowledge. If students are comparing 4/4 with 4/8, fraction bars will force them to the right answer that 4/4 is more than 4/8:



They do not have to consider directly the role of the common unit, for it is implicit within the material. Yet, if students construct their own models, they may confront and have to struggle with this essential concept, as one nine-year-old did when he drew, at first:



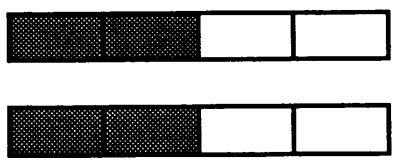
This drawing made it seem as though 4/4 might be equal to 4/8 and he and his classmates struggled with the question of whether the rectangles had to be the same size in order to compare two fractions. One classmate asserted that they did, because otherwise, "your drawing would convince you of something that wasn't true——four-fourths is really *more than* four-eighths." Another student, however, argued that it didn't really matter how big you made the rectangles because you could see that 4/4 took up all of the rectangle, while 4/8 took up only half of it. This valuable discussion would probably have never come up if the students were using fraction bars. Fruitful representational contexts are framed clearly enough to facilitate the development of sound mathematical understandings and skill in students. Fraction bars, pie diagrams, number lines——all these can help to focus learners on certain key features of fractions, such as the meanings of fractional terms. At the same time, the context is sufficiently open to afford students opportunities to explore——to make conjectures and follow important mathematical tangents. The example above suggests that there are times for letting learners confront and grapple with conceptual complexity (cf. National Council of Teachers of Mathematics, 1989b). Managing a suitable tension between focus and openness in the representational context is crucial. **Considering students and how they learn**. Beyond mathematical considerations, another layer of contemplation emerges in considering what students understand and how they learn. Nesher (1989) points out that "the child should be familiar with the exemplifying objects and be able to use familiar language to describe and communicate relations among these objects" (p. 194). Certain representational contexts, although mathematically reasonable, are nevertheless inaccessible to students (Dufour-Janvier, Bednarz, and Belanger, 1987). For example, although electrical charges may provide a mathematically promising model for the multiplication of negative numbers, sixth graders are as unfamiliar with the behavior of electricity as they are with the behavior of negative numbers. As such, electricity will not make an accessible representation for teaching about negative numbers. Other representational contexts, while engaging and accessible to students, are mathematically distorting or thin. For example, the everyday idea of borrowing may distract students from regrouping and place value two-digit subtraction, and may encourage them to think of numbers in the right hand column "borrowing" equal-sized numbers from the next column.[4] **Putting representational contexts into use**. Representational contexts are not static and do not stand alone. They offer "thinking spaces" for working on ideas. In order to be viable and useful, these thinking spaces must be furnished and developed jointly by teachers and students. Language, conventions, and other mental props are necessary. For example, although money and debt may seem——to adults——potentially helpful in making sense of negative numbers and operations on the integers, nine-year-olds may not be inclined to reconcile debt with cash to obtain a figure of "net worth." Rather than reporting a balance of -$4, my third graders were disposed to report that "so-and-so owes his friend $6 and also has $2 in his pocket," thereby avoiding using negative numbers at all. Thus, exploiting the representation successfully requires figuring out conventions for its use. I worked to find language and stories that would encourage students to represent debt differently from money——and to want to reconcile the two (see Ball, 1990a).

Similarly, the third graders described above had to construct conventions and language for using rectangles (which were often representations of brownies or graham crackers) to represent, compare, and operate with fractions. To represent fractions, they developed strategies for making the drawings: Sean conjectured——and others agreed——that "to make some number of pieces, make one less line." In other words, to make thirds, draw two lines in your rectangle. Acknowledging that no one could draw perfectly equal pieces, the children had to agree how fussy to be about the pieces looking equal. They also struggled with whether the rectangles had to be the same size in order to compare them, and

[4]In Ball (1988), I describe how prospective teachers trying to find representational contexts for teaching about regrouping actually thought "borrowing" was a fruitful representation for subtraction because children would be familiar with borrowingfrom neighbors. See below for a discussion of learning to deliberate about representation in pedagogically defensible ways.

what it would mean to try to combine two different fractional quantities. Real-world concerns sometimes collided with the mathematical viability of the representation. For example, is 3/3 greater than, less than, or the same amount as 5/5? Some children argued that 3/3 was more because each piece (one-third) was bigger. Others argued that 5/5 was more because there were more pieces. Still others thought that they were the same because each represented one whole brownie. For these rectangles to offer a fruitful thinking space for children to explore fractions, the representation must be embedded with agreements about what "more" or "greater" means——that it is the total quantity, not the number or size of the pieces.

These thinking spaces are broadened——and the accompanying issues expanded——when multiple representational contexts are used for a given topic. Teachers and students must work through the links among them and how one moves from one to another. For example, using the number line to compare 3/3 with 5/5 presented few problems: The two were obviously the same. But how that relates to rectangle drawings is not a straightforward matter for learners. If students conclude, using the number line, that 3/3 is the same amount as 5/5, they may still think that one is more than the other when using rectangle drawings. Similarly, some children decided that 2/4 + 2/4 = 4/4, or 1, when they work with the number line——but that it equals 4/8 when they use a regional model:



This conclusion arises, not out of a failure of the representation itself, but from lack of agreement about how to use it. The students who argued that this drawing showed that 2/4 + 2/4 = 4/8 reasoned as follows: There are eight pieces total and four of them are shaded. This representation matched the students' assumption that, to add two fractions, one would add the numerators and denominators——a fact that only reinforced their conviction that what they had done made sense. To reason about addition of fractions using such area models requires that one agree to hold the unit constant (Leinhardt and Smith, 1985). If the unit is one rectangle, then 2/4 of one rectangle and 2/4 of another rectangle will fill up one whole rectangle, or 4/4. The students who believed the answer to be 4/8 were looking at *two sandwiches* as the unit.

The conventions, language, and stories that support the use of a given representational context are crucial to building valid understandings and connections. In this case, the teacher could pose a story situation that would provoke students to consider the importance of maintaining the unit——for example: "Marta ate 2/4 of a sandwich at noon and 2/4 of a sandwich after school. How much did she eat?" Students might be able to discuss that she ate the equivalent of one whole sandwich or four quarters of sandwich. They could also discuss the notion that she *has* eaten 4/8 of *two sandwiches*—— and thereby reach some agreement on the importance of identifying the unit——and of choosing a useful unit.

Teaching as inquiry. Teaching is essentially an ongoing inquiry into content and learners, and into ways that contexts can be structured to facilitate the development of learners' understandings. Representations are conjectures about teaching and learning, founded on the evolving insights about the children's thinking and deepening understanding of the mathematics, and one must inform the other in the construction and use of representational contexts. In this paper, I examine the pedagogical thinking and work involved in understanding, constructing, and exploiting representational contexts for learning mathematics. My thesis is that deliberating about the construction and use of such contexts is at the core of teaching mathematics for understanding. Finely tuned analysis of the content, as well as rich knowledge about students and how they make sense of that content, can and should play a central role in teacher thinking and practice.

To illustrate some of the complexities in thinking through and using representations of mathematical ideas, I will draw examples from my own teaching. Using myself as the object and tool of my own inquiry within and about teaching mathematics for understanding, I teach mathematics daily to a heterogeneous group of third graders at a local public elementary school. Many students are from other countries and speak limited English; the American students are diverse ethnically, racially, and socioeconomically, and come from many parts of the United States. Sylvia Rundquist, the teacher in whose classroom I work, teaches all the other subjects besides mathematics. She and I meet regularly to discuss individual students, the group, what each of us is trying to do, the connections and contrasts between our practices. We also spend a considerable amount of time discussing and unpacking mathematical ideas, analyzing representations generated by the students or introduced by me, and examining the children's learning.

Every class is audiotaped and many are videotaped as well. I write daily in a journal about my thinking and work, and students' notebooks and homework are photocopied. Students are interviewed regularly, sometimes informally, sometimes more formally; sometimes in small groups and sometimes alone. We have also experimented with the methodology of whole-group interviews. I give quizzes and homework that complement interviews and classroom observations with other evidence of students' understandings. This paper draws on data from my teaching of fractions during 1989-90.[5]

Among my aims is that of developing a practice that respects the integrity of both mathematics as a discipline *and* of children as mathematical thinkers (Ball, 1990b). I take a stance of inquiry toward my practice, working on the basis of conjectures about students and understandings of the mathematics; in so doing, both my practice and my understandings develop. This paper traces my struggles to engage third graders in developing their understandings of fractions. My deliberations about my teaching of fractions serve to illustrate dimensions important to the pedagogical reasoning that underlies the engagement of students in representational contexts.

[5]Currently, Magdalene Lampert and I are engaged in an NSF-funded project to produce and explore the use of hypermedia materials in teacher education (Lampert and Ball, 1990). Our aim is to construct a representational context for *learning to teach* in which teachers would develop new ideas and ways of thinking, new questions and things to consider, and new senses of problems of practice and ways to work on them. Just as in teaching elementary mathematics, where our goal is to engage students in significant mathematical inquiry, a representational context for learning to teach grows out of our conception of *teaching practice* as inquiry. We want to provide a terrain in which teachers can explore and investigate as well as acquire tools for their investigations (e.g., deeper understandings of mathematics, new perspectives on children as learners, new ideas about curriculum and the teacher's role). Hypermedia technology is promising for the design of such a representational context. This work with Lampert has contributed significantly to my thinking about pedagogical reasoning in mathematics.

The Construction and Use of Representational Contexts:
Pedagogic Contemplations on Fractions and Third Graders

What representational contexts can help third graders construct useful and sensible understandings of fractions? In deliberating about this, two concerns are prominent: *subject matter*——what students should learn about the territory of fractions——and *learners*——what students already know and how they learn.

Rational numbers is a domain in which there has been considerable work and detailed analysis (e.g., Behr, Harel, Post, and Lesh, in press; Behr and Post, 1988; Kieren, 1975, 1988; Nesher, 1985; Post, Behr, Harel, and Lesh, 1988). Among the analyses, some agreement exists that fractions may be interpreted (a) in part-whole terms, where the whole unit may vary; (b) as a number on the number line; (c) as an operator (or scalar) that can shrink or stretch another quantity; (d) as a quotient of two integers; (e) as a rate; and (f) as a ratio. Nesher (1985) also includes fractions as representations of probabilities.

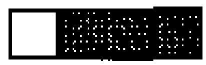
In my journal, I worked on a conceptual map of fractions——the constructs entailed and the connections between fractions and other important mathematical ideas. As I considered the multiple senses of fractions, the relations between fractions and division, multiplication, measurement, functions, probabilities, numeration, and so on, the complexity of the topic emerged. As Ohlsson (1988) observes:

> The difficulty of the topic is . . . semantic in nature: How should fractions be understood? The complicated semantics of fractions is, in part, a consequence of the *composite nature* of fractions. How is the meaning of 2 combined with the meaning of 3 to generate a meaning for 2/3? The difficulty of fractions is also . . . in part, a consequence of the bewildering array of *many related but only partially overlapping ideas* that surround fractions (p. 53).

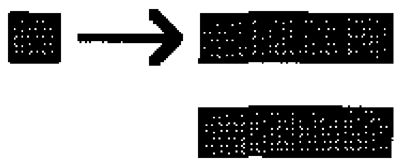
In thinking about the content, I also examined the state and school district objectives for fractions. The state objectives (on which my students are tested at the beginning of fourth grade) require students to be able to match fraction symbols with area models for halves, thirds, and fourths——for unit fractions only——and that they be able to identify congruent parts. The school district's objectives include recognition of 1/6 and 1/12 but, like the state objectives, they also deal only with unit fractions. Students must be able to identify the number above the bar as the numerator and the number below the bar as the denominator and be able to multiply a whole number by a unit fraction (e.g., 1/2 x 6). Students should also, according to these objectives, develop "an understanding of the meaning of fractions." Unlike the other objectives, this one is a mouthful, given the breadth of meanings that can be assigned to fractions. In third grade, I currently focus on the first three interpretations and applications of fractions:

1. **part/ whole** -- the description of dividing a given unit (quantity) into some number of parts:

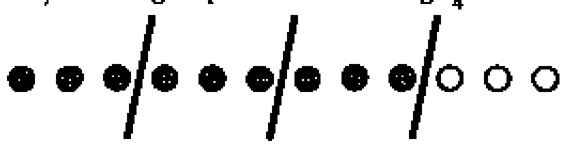
   - Taking 1 to be the unit and dividing it into some number of parts and taking some number of those parts (e.g., $\frac{3}{4}$):



   - Taking 1 to be the unit, dividing it into some number of parts, and taking some number of parts *of that size* (e.g, $\frac{8}{4}$):



   - Taking some other number of objects to be the unit and dividing the set of discrete objects into groups of some size (e.g., $\frac{3}{4}$ of a dozen)



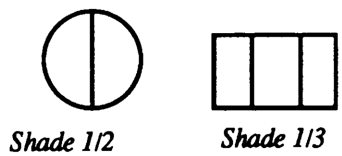
2. **linear coordinate**: fraction as a *number*, as a point on the numberline

3. **operator** -- fraction as something that *operates on* and shrinks or stretches another quantity

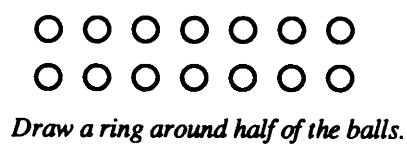
   shrinking: $\frac{1}{2} \times 6 = 3$ *stretching:* $\frac{5}{2} \times 4 = 10$

Proportional reasoning and intuitive use of odds also come into play as we explore probability. Aware of the breadth of the topic of fractions, I am cognizant of how my choices may limit or constrain the horizons of their mathematical trajectories. Uncertain about my decisions (cf. Floden and Clark, 1988), these are open to ongoing reconsideration and revision.

In addition to contemplating the content, I also considered what nine-year-olds may have previously encountered about fractions——in school and out. My familiarity with the district curriculum and with a range of curriculum materials told me that, in school, they would likely have had limited experience, consisting primarily of shading predivided regions, such as:



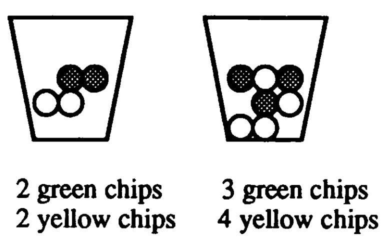
They probably had not had any experience dividing regions themselves and it was quite possible that they would have had no experience with any non-unit fractions (i.e., fractions with numbers other than 1 in the numerator——2/3, for instance). Possibly they would have examined fractional parts of discrete sets, for example:



The fractions examined with discrete sets would probably have been halves, fourths, and possibly thirds. I was quite sure that they would not have dealt with anything other than unit fractions: that is, they probably had not had to figure out how many balls were in two-thirds of the set.

I also considered what I had learned about my students' ideas and thinking from our work in related topic areas, such as probability. In that context, the students had not formally quantified probabilities as fractions, but they had compared the likelihood of particular events. For example:

From which cup is it more likely to pull a green chip?



Thinking about this problem did entail proportional reasoning. When I designed it, I was keenly aware that it would simultaneously push the children *and* help me learn about their intuitive fraction knowledge. Problems such as that are deeply useful to me as I wend my way across the terrain of third-grade mathematics with a particular group of learners (cf. Lampert, in preparation).

I knew in this case that the problem had the potential to press the children to consider the numbers of both green *and* yellow chips in order to answer a question that appeared to be only about the green chips. Without representing the probability of pulling out a green chip from either cup, what was key was recognizing that the answer lies in paying attention to the ratio of green to yellow chips in each cup. I had made notes in my journal about the ways in which different children reasoned about problems such as this. Some students, for example, reasoned that pulling a yellow chip was more likely from A than from B because "there are more yellows than greens in cup B, so yellow is more likely than green, but in cup A they are equally likely." Some students' patterns of reasoning did not consider the multiplicative structure of the problem and argued that it was more likely to pull a green chip from cup B because there were three greens chips in cup B but only two in cup A.

During the probability unit, students had repeated experience with such questions and arguments, although no effort was made to record probabilities symbolically. Thus they never talked about the probability of pulling a green chip out of cup B as 3/7. Still, our work in this area informed my understanding of the students' proportional reasoning in ways that were helpful as we began our more formal foray into fractions.

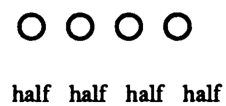
Beyond school, I also reflected on what I knew about their out-of-school experiences with fractions. They likely had everyday experience dividing things in *half* but perhaps not in thirds, fourths, or fifths. They probably had experience with money——especially quarters. Most would be comfortable telling time to the quarter and half hour. Many would have used fractional cup measures when baking or cooking. Across these contexts, I suspected that their concept of unit would be strongest with money, where they would know that a quarter was 25¢, a half dollar, 50¢, and a whole dollar, 100¢. With money, they understood that there were four quarters and two half dollars in one dollar.

With time, I was less certain what they understood explicitly. Even if they were familiar with quarter past, quarter to, and half an hour, many did not know how many minutes there were in a quarter hour or a half hour. Nor was I certain they would know why we speak of quarter hours, that is, that an hour has 60 minutes and that a quarter of an hour is *called* a quarter because it is 15 minutes, which *is* 1/4 of 60 minutes. From baking experiences, I knew that with measuring cups, the children used "quarter" or "third" or "half" as names for the different size cups, rather than as proportional or relational terms. For the most part, they did not know to expect that there would be four quarter-cups in a whole cup.

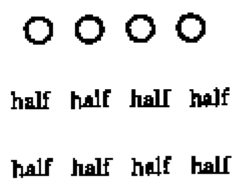
What did I know about my third-graders' understandings based on these experiences? I saw that their fraction knowledge was scattered across a range of in- and out-of-school contexts, their understandings situated in particular uses (Brown, Collins, and Duguid, 1989). Most also had some generalized understandings. For example, their understanding of one-half tended to be quite robust: they were able to consider both one-half of a whole unit and one-half of a set. Many were able to locate 1/2 on a number line and most could record something like 1/2 to represent one-half. At the same time, many of them also generalized one-half to apply to *any* part of a whole. Harooun,[6] for instance, when working on the problem, "How much can each person have if there are four people trying to share five brownies equally?," offered additional evidence of this way of thinking. He drew four people:

and five brownies

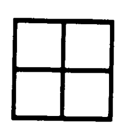
He explained that he divided each brownie in half and gave everyone a half, which he recorded as:



He repeated this, adding another half to the pile of brownies for each person:

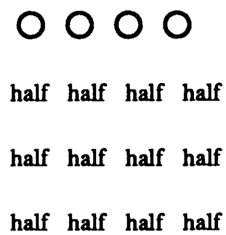


Then he took the last brownie and divided it into "little pieces":



[6]All names used are pseudonyms and are drawn appropriately, to the extent possible, from the individual children's actual linguistic and ethnic backgrounds.

and recorded these "little pieces" as halves, too:



Harooun's conclusion was that the four people would each get "three halves." In the discussion, the other students agreed with Harooun's solution. Some children disputed his labeling the little pieces halves. Eventually, there was consensus that, because he had divided the brownie into *four* pieces, these were *fourths*. This seemed to be a new, but sensible, term to the third graders. I had seen little evidence that third graders had anything other than a fragile, schoolish knowledge of thirds and fourths. Fifths, eighths, tenths, and so on, were basically unfamiliar and their understanding of halves, thirds, and fourths did not tend to set up the construction of other fractions. "Half" was more a quantitative habit of mind than an explicit concept.

Their ways of thinking about number based on their immersion in whole numbers was another source of insight for me. Again, from my notes on earlier, mathematically related work, I knew that they assumed that the number line represented the number system, a set of discrete points, that there were no numbers between the dots. The next number after 1 was 2, and after 2, obviously 3. They also were in the habit of thinking simply of numbers as representing quantities: "two" could refer to two pencils, or two shoes, or two cookies.

As we worked on multiplication, they began to have experience with essentially multiplicative units other than one——dozens or weeks——for example. That "two dozen" referred to 24 objects was a difficult idea for some of them, for until now, two had meant, quite simply, two single things. When we began talking about fractions, they tended——as Harooun did above——to speak only of the number of pieces, irrespective of the partitioning units. By this way of thinking, some thought that 2/5 was more than 1/2 because there were more *pieces* (i.e., two pieces shaded in 2/5 and only one in 1/2).[7]

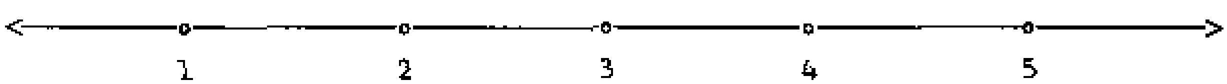
[7]Mack (1990) reports that fourth graders in her study were able, informally, to compare one-sixth to one-eighth. However, when they are presented with symbols and asked to compare 1/6 and 1/8, they invoked their assumptions about whole numbers and said that 1/8 was more——because 8 is more than 6. Here my third graders were being asked to compare fractions where the numerators differed as well; this suggested that the "number of pieces" strategy led them to incorrect conclusions with non-unit fractions.

Content analyses of the domain of rational numbers (e.g., Behr and Post, 1988; Kieren, 1988) and research on children's thinking about fractions (e.g., Lefevre, 1986; Larson, 1988; Mack, 1990; Tierney, 1988) gave me lenses for watching my students, cues for listening to them. It was in working with the third graders, however, that the most crucial issues of content and learning emerged for me. Looking at rational numbers from the perspective of a nine-year-old whose familiar mathematical domain is being stretched and transformed, I saw aspects of rational number thinking that I had not noticed before. For example, I realized that making sense of unit fractions (e.g., 1/3, 1/4) requires only one part of the thinking entailed in comprehending fractions: To understand what 1/4 means, one need only divide the whole into four parts. Non-unit fractions require complex compositional thinking: 3/4 entails both dividing into four parts and multiplying the result by 3. I realized, then, that when children have worked only with unit fractions, they may not be confronting the essentially compositional nature of fractions.

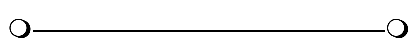
They had little experience making sense of the written notation of fractions. While they quickly began to read fractions correctly, they were unsure about what they meant. For example, Mei interpreted 3/4 as "make groups of three" and then take all but one group (i.e., 3/4 is one less than 4/4)

They also tended to have imagic (visual) rather than principled knowledge of familiar fractions; for example, that 1/4 is *this* shape:

Even when we worked with the number line as a representational context for exploring fractions, it turned out that some children conceived of one-fourth as a fixed unit of measurement, like a centimeter, such that it would be impossible, for instance, to represent fourths on this number line:



because one-fourth is *this* (apparently arbitrary) length:



Dividing a quantity into four equal parts has little to do with this way of understanding one-fourth; the understanding is more visual than conceptual.

The ways in which numbers function as scalars was also highlighted for me as I discovered——through the students' eyes——the essentially relational and referential nature of fractions. With whole numbers, 5 may mean "five" if the referent is *one object* but sixty if the referent is *one dozen*. For children, at least, this idea emerges more prominently as they engage fractions. One-fourth may mean 25 (as in "one quarter of a dollar") or 4 (as in "one quarter of a pound"). Third graders are able to reason comfortably with one-half (i.e., they can think flexibly about 1/2 of a dozen, a dollar, a yard, one cookie), but their ideas about other fractions assume fractions of one——or fixed units of some other size (e.g., for some children, separating the idea of "one-quarter" from the coin is problematic).

This came through most vividly to me one day when we were discussing solutions for the problem, "What is 1/4 of a dozen?" Several people argued that it had to be 4 (misconstruing the meaning of the 4 in the denominator). Other saw that it was 3 and they managed to convince the rest of the class of their solution——except for Lindiwe. His objection, as he voiced it, was, "How can 3 be one quarter of a dozen when one-quarter is just a little piece?" and he went to the board and drew:

Lindiwe's misconception underscored my sense that, for some nine-year-olds, in spite of the fact that they often do get the right answers on school fraction tasks (e.g., "Shade one-third"), their understandings of fractions may not be principled, but are based instead on remembered images. For Lindiwe——and for some of his peers——the little wedge *is* one-fourth.[8]

The Joint Construction of the Representational Context for Learning Fractions

In my struggles to create and orchestrate fruitful representational contexts in which my students could explore mathematical ideas, I have come to see that representational contexts are co-constructed and developed by members of the class. Students enter the representational context that the teacher has set up and, in dealing with a specific problem, they generate alternative ways to represent or check their understandings. Together, students and teacher must develop language and conventions that enable them to connect and use particular representations in situations. They must also develop ways of reaching beyond and across specific situations to abstract and generalize emergent understandings. The representations are tools to be wielded in mathematical investigations——in framing and solving problems, in making and proving general claims. The tools themselves are sharpened and developed through these processes. Students also sometimes invent or introduce representations independently.

[8] This is similar to children's visual approach to geometric objects. Squares are typically not permitted in the category of rectangle, for rectangles must have "two long and two skinny sides"——exactly what they have seen in workbooks.

The following case from my teaching of fractions illustrates this joint construction of the representational context. My work with my students over this is also a good illustration of the pedagogical dilemmas entailed by the horns of Nesher's (1989) dilemma: that, on one hand, students must construct their knowledge through interaction with the environment *and* that, on the other hand, teachers are responsible to help students develop *particular* mathematical ideas.

As we were moving from division toward fractions (on a voyage that parallels the emergence of fractions in the history of mathematics), I presented the class with the following problem:

You have a dozen cookies and you want to share them with the other people in your family. If you want to share them all equally, how many cookies will each person in your family get?

I conceived this problem as a thinking space in which I hoped to stimulate students to develop several key understandings of fractions. I used it on a cusp between an extended period of explicit work on multiplication and division (which had involved fractions) and the beginning of some direct work on fractions (which would continue to involve multiplication and division). The problem involved the partitive interpretation of division (forming a certain number of groups) and would produce multiple solutions.

For some size families, there would be cookies left over which could be divided further. Based on what I knew about the families of my students, I realized that we could encounter fifths, sevenths, and probably both halves and eighths. I also knew that students would probably be inclined to divide the leftover cookies, but would not necessarily know what to call the pieces they produced. Still, the children would probably see fifths and halves as clearly different in amount, hopefully motivating a need to name pieces in meaningful ways. I anticipated, in short, that this problem would launch us into an extended exploration of fractions.

First we had figured out how many cookies everyone in *my* family——with four members—— would get. Then the students worked independently or in pairs or threes to figure out how the dozen cookies would work out in *their* families.

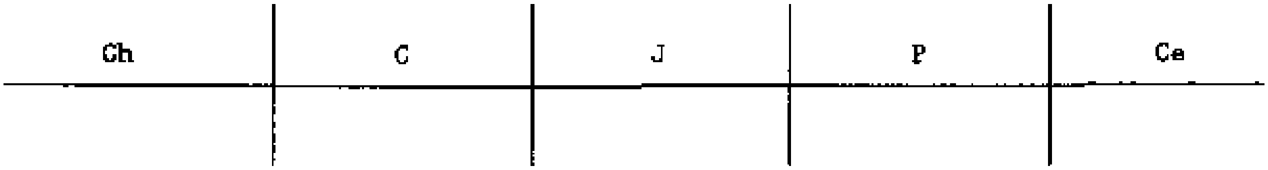
I heard some discussion about whom to count as a member of one's family. Keith wondered if he should count his about-to-be-born baby brother or sister while Riba decided *not* to count her new baby sister ("She can't *eat* cookies!"). Sean noted that "my dad doesn't like cookies" and did not include him. I was also uncomfortable as I overheard some students questioning other students' counts. Mei asked Lucy, "Who's the fourth person? You only have three people in your family." Lucy, matter-of-factly, responded that she was counting her mother's boyfriend who was living with them. Someone else challenged Lindiwe's counting his father since his parents were divorced and his dad was currently living in Washington, DC.

These conversations seemed intrusively personal and I found myself questioning my decision to contextualize the problem in terms of families. I had done this because the divisor would vary nicely among the students, allowing for a range of interesting solutions, some simpler than others. I knew we would end up discussing division of 12 by 2, 3, 4, 5, 6, and 7——and that 5 and 7 would lead us into fractions: my destination. This was exactly where I now wanted to move from our work with division and multiplication. But, as I listened, I questioned my choice, for the goodness of a representational context depends on its social and cultural appropriateness as well as on content and learning factors. I decided to discuss the issue with the class the next day——to ask them what *they* thought about the problem and the interactions that surrounded it.

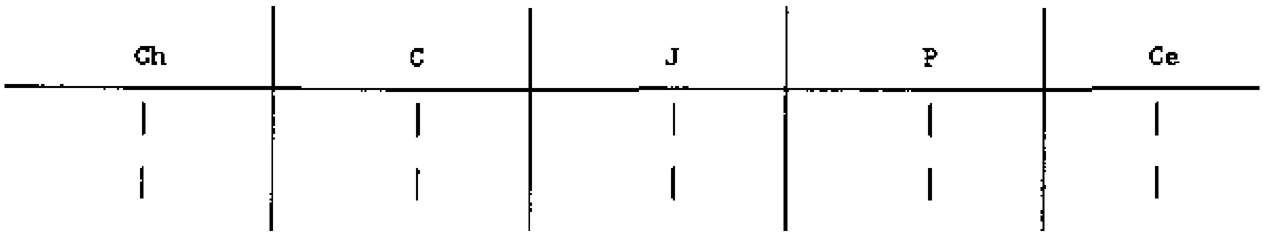
In this discussion the next day, many children said that the problem seemed okay to them, that they had not minded the questions that came up around it. Betsy, however, empathized with how some students might have felt: "Well, for some people I think it would be sort of being nosy, because if somebody really missed their dad and they didn't want people talking about it, that would make them feel even sad or something like that, so it might not be such a good idea." Tory agreed. At this, Lindiwe spoke up and said that many people kept arguing with him, saying that he only had four people in his family and he kept explaining that he was counting his dad. I asked how he felt about that and he said that he liked the discussion of the problem but that he thought people should let him decide whom he wanted to count in his family: "I think that people shouldn't really be saying how much you have in your family. They don't know because they've never been to your house. So, they shouldn't really tell you stuff that they don't even know." After listening to their comments and thinking about the problem myself, I thought I would not be inclined to use this problem again——at least in this particular context——for it seemed too intrusive and potentially personal despite the fact that the problem had both believability and significance.

After I posed the problem, I had walked around the room, listening and watching. Most children were working in pairs or threes. A few were working alone. During this work period, I try to learn how different children are thinking and how they are interacting with the representational context I have framed. I ask questions, sometimes playing devil's advocate, sometimes pressing for clarification, explicitness, or depth. Sometimes I encourage them to confer with a classmate. Sometimes I provide a piece——either information or a question——to spark or spur further thinking. This phase of the class period is crucial to the joint development of the representational contexts in which we are working, for it is a primary source of information about what the students are thinking and how they are making sense.

Cassandra, with five people in her family, was working at the chalkboard and was eager to show me her work on the problem. Adding her own representation, she had drawn a chart as a tool for and display of her reasoning:

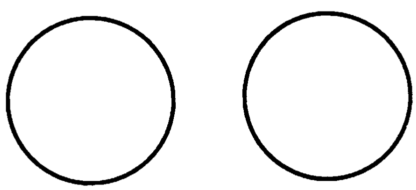


The letters in the columns, she explained, were the first initials of her family's names. Then she distributed 10 of the cookies by making hash marks across the columns until each member of her family had two hash marks, representing two cookies.



Cassandra: Um, I would have 2 cookies left over so I figured what I would do with those 2 cookies? I would split them in half or either just throw them away.

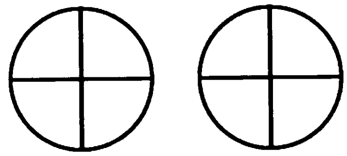
(She drew two circles on the board):



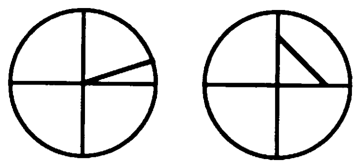
So here's two.

She drew lines in the circles, cutting them first in half and then in quarters and described what she was doing:

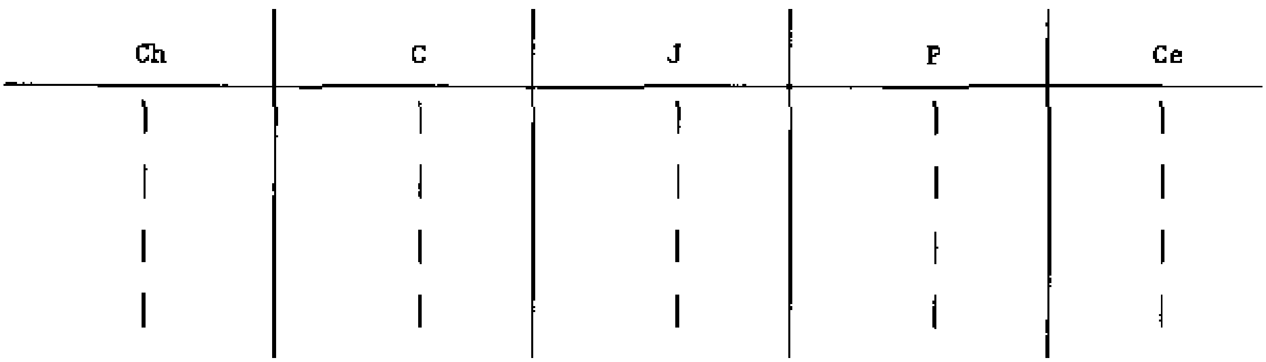
I cut them in half and then in half again and so there's four.



But I have 5 people in my family, (adding another line to each cookie) so there's one more.



And Cassandra added two more lines for each person on her chart:



Then I asked how many cookies she would give each person in her family. Cassandra counted the hash marks: 1, 2, 3, 4.

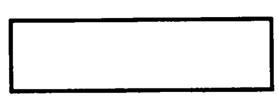
Cassandra's solution was intriguing. On one hand, she got a close approximation of a "right" answer——(2-2/5). On the other hand, she reported it as 4, counting *pieces* irrespective of size. In most classrooms, Cassandra's solution would be judged to be wrong. After all, her conclusion in writing was 12 ÷ 5 = 4. Even after looking at her cookie drawings——which may, in fact, represent 2-2/5—— questions remain about Cassandra's intuitive understanding of fractions. She realized that the five pieces (inside each of the two leftover cookies) are not the same size. Did she mean them to be equal but just did not know how to *draw* fifths properly? Dividing a circle into five equal parts is no easy task. Or did Cassandra not recognize that equal size is a crucial aspect of dividing something like cookies equally? Was she focused only on coming up with the same number of *pieces*?

A "number of pieces" frame makes sense in many integer-division contexts: sharing a bag of different lollipops, a box of assorted pencils, a pile of books, or a sack of marbles, for example. In such cases, the collections would probably be considered to be divided equally if each person got the same number of items. The idea that sharing a quantity equally involves an equal division of its mass arises much more prominently only in contexts where items will be subdivided into fractional parts. Thus, for Cassandra and her classmates, that fractions implied equal parts was not necessarily obvious. At this point, as we began our work on fractions, the centrality of *unit* was not obvious either. Mack (1990) reports similar results in her investigation of fourth graders' informal knowledge of fractions: Students focused on "breaking fractions into parts and treating the parts as whole numbers rather than as fractions" (p. 28). That evening, I wrote in my journal:

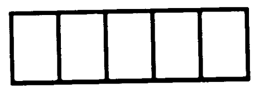
> One interesting thing to me about her clever solution was that, contrary to what I've tended to assume, Cassandra did not seem to focus on the pieces being "fair"——i.e., equal in size. What mattered more, it seemed, was having the right *number* of pieces. Is that an artifact of the representation? If she was dealing with real cookies, would she deal with it in the same way? I remember some arguments from last year's class when the "number of pieces" frame dominated so that 4/8 and 4/16 seemed the same to some people.

In class, after listening to her solution, I debated about how to respond to Cassandra. Should I question her further about her solution? She was not at all dissatisfied with it and it made compelling sense in many ways. Yet I thought I saw an opportunity to respect her genuine attempt to distribute 12 cookies among the five members of her family *and*, at the same time, extend her thinking by helping her develop some new tools to accomplish that goal.

I saw that the fact that the problem entailed cookies encouraged the use of a circle representation——an unfortunate obstacle, since drawing equal parts inside a circle is technically difficult. This difficulty makes it harder to determine whether a child *intends* to divide the circle equally——and just does not know how——or whether the child is even considering the importance of equal parts. I decided to adjust the representational tool and suggested to her that we draw rectangular cookies. It would be easier, I said, to divide them up equally so that everyone would get the same amount of cookies. Because I wanted to make sure that the problem remained well-connected to some real situation for Cassandra as we shaped the context together, we talked for a moment about kinds of cookies that are shaped as rectangles: hermits, windmill cookies, and brownies. Then I drew:



and asked Cassandra to divide up the cookie for her family. She drew four lines, counting the now-equal pieces: One, two, three, four, five:



Cassandra wanted to call these pieces "halves." The terms we use for fractional parts is a matter of convention, not invention (Lampert, 1990; Larson, 1988). Cassandra would not discover, on her own, what to call her pieces. I told Cassandra that we call those parts not "halves," but "fifths." Then I asked her if she could think of a reason why that made sense. She quickly replied that it made sense because the cookie had been divided into *five* pieces. I showed her that the way we write "one-fifth" looked like this: 1/5——again, conventional knowledge. She said that made sense because we had divided it into *five* pieces and one-fifth was *one* of them.

I asked Cassandra if she could divide up the other leftover cookie. She did this. Then we talked about how much cookie someone would get if they got one piece from each of the leftover cookies. Looking at the two cookies that had been divided into fifths, Cassandra realized that each person was to get 1/5 and another 1/5. Concentrating on her new understanding of something called "fifths," she appeared to be thinking with the symbols, rather than from her pictures. Cassandra appeared to abandon her more conceptual, pictorial approach and began thinking in a symbolic mode. Thinking of the denominators, she began, "5 + 5 is __." I prompted, "No, think about your picture. One-*fifth* plus one-*fifth*." She paused to think about this, and then said "two of the fifths." Cassandra's inclination to rely on the symbols fits with Mack's (1990) finding that fourth graders' "isolated knowledge of procedures . . . frequently interfered with their attempts to give meaning to fraction and procedures" (p. 27). Rather than thinking intuitively about what it might *mean* to add one fifth and another fifth, Cassandra switched over to thinking about adding the numbers in the symbolic form.

I was thinking about what Cassandra understood and how we had together shaped the representational context, but Mei was tugging at my sleeve to come and see what she and Tory had done. I listened to their solutions, still gathering information about how the children were working within the representational context. Our time was almost up. I could tell from scraps of conversation that we were ready for a group discussion of the problem. I left Cassandra, asking her to try to figure out how much every member of her family would get now.

The next day, I opened the group discussion of the problem by asking for volunteers to give their solutions. Jeannie explained her solution for three people in a family; Maria agreed with Jeannie's answer and showed a different way——using a picture——to prove that three people would each get four cookies. There was no disagreement; several students said they agreed with both Jeannie and

Maria.

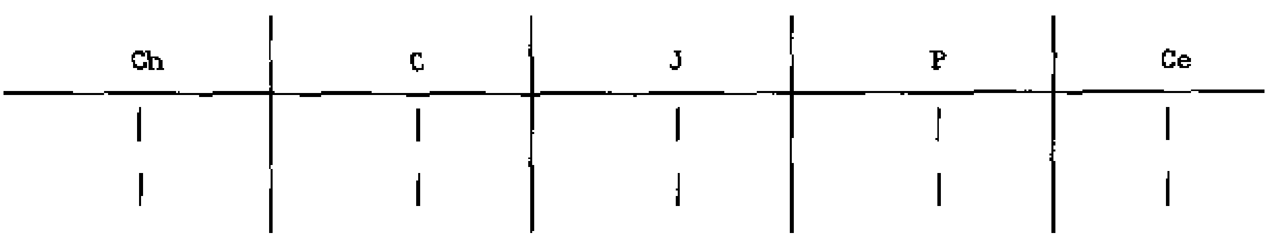
I suggested that we next discuss solutions for two people in a family. Then we moved on to five. I knew that, in addition to Cassandra, Riba, Daniel, and Sean had also been working on solutions for five people in a family. Riba said she was still working on it, that Cassandra should present her solution. I was curious in seeing whether and how the group context would affect Cassandra's current thinking about the problem. We had worked hard at creating a classroom culture in which it was safe to try out an idea that you did not yet have full hold of, that you were unsure about, that was fragile. Now in the middle of the year, the students had grown to be quite respectful of one another's thinking and were patient with stumbling explanations. They were also inclined to ask questions to understand how a classmate was thinking before they suggested revisions or disagreed with an idea.

I wondered whether presenting her solution to 12 ÷ 5 would help Cassandra to strengthen her understanding of the problem——that her thinking would be clarified through what she would have to think about in order to explain her solution to the others and through the questions others might ask. I wanted to see whether, with support from me, if necessary, she could show what she had done, and get the other students to appreciate the thoughtfulness and sense of her solution. The complexity of the problem and its solutions would tilt the class toward fractions, the direction I wanted to head.

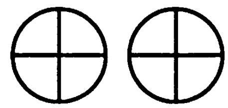
Cassandra went to the overhead and, leaping over the first part of her solution (that each person could get two whole cookies), she drew two circles——the leftover cookies. Hoping to push her gently, I intervened:

Ball : Cassandra, are you going to use your rectangular cookies?

Cassandra : Uh huh . . . Okay, so alright——(*and she backed up to the beginning of the problem and made the chart she had made on the board when she was working alone earlier*)——here's my sister, my brother, my dad, and my mom. Okay, and I have 12 cookies, so (*distributes the cookies, making green hash marks on the chart*) 1, 2, 3, 4, 5, 1, 2, 3, 4, 5 . . .

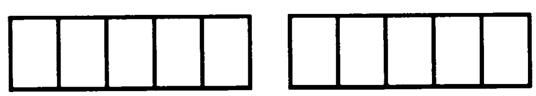


I have two cookies left over so what I do is draw two cookies . . . I divide them, you all got two cookies apiece so 1 here, 2, 3, 4.



I debated: Should I let her pursue this, dividing the cookies only into fourths and have other children argue with her? The group was able to work well to negotiate what makes sense. But Cassandra would also often tenaciously maintain her point of view. She would also sometimes falter, erase, and abandon her presentations when a flurry of questions arose. Wanting to both press her thinking a bit and keep her at it, I decided to support the new explanation instead: I was curious about what role presenting it to others would have. I also wanted the idea on which Cassandra was verging to become part of the group's working knowledge (Edwards and Mercer, 1989). I reminded Cassandra that she needed five pieces, not four, and that working with rectangles was easier.

Cassandra : They each got two cookies (pause) draw the other two that was left over from the 12 . . . put five lines, so it . . . here's one cookie there, put 2, 3, 4, and 5. Then the same here, 1, 2, 3, 4 and 5 so . . .



Cassandra put two more hash marks under each person's column on her chart, but used orange marker instead of green this time. These orange marks represented the fifths as distinct from the whole cookies, progress from yesterday's work when 2-2/5 seemed, to her, to be 4.

Ball : So how much did each person get?

Cassandra : Two-fifths.

Ball : (pointing at the chart) What are those green marks? What kind of cookies are those green marks?

Cassandra : Two whole.

Ball : Two whole cookies and the orange marks are——?

Cassandra : Two-fifths.

Ball : Two-fifths cookies. Okay. Comments and questions for Cassandra?

Turning the discussion over to the students is a typical routine in our discourse. In trying to help students develop the capacity to determine for themselves whether something makes sense mathematically——rather than relying on the teacher or the text (cf. Lampert, 1989), I deliberately structure our discussions so that students respond to one another's ideas, comments, and solutions.

The other students seemed to think that what she had done made sense. To press the students' understanding of the importance of unit, I asked why Cassandra didn't say that each person would get four cookies——there *were* four hash marks under each column. Temba said he wasn't sure, showing me that this was not obvious to everyone. Tory said that "they're split in *half* so that wouldn't be four cookies because they're not whole cookies." Others nodded.

I was both pleased and concerned with Tory's answer. On one hand, she was recognizing that the unit was changing and that you could not count both whole cookies and "half-cookies" as wholes: two whole cookies and two pieces was not four. On the other hand, she was also still referring to any part of a whole as a "half," a common habit among the third graders. I asked for comments on what Tory had said.

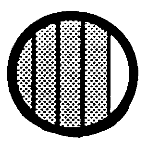
Mei said that she agreed. I pushed: Did anyone have an idea why those little orange hash marks weren't called "halves"? Lucy explained that halves were bigger than fifths. My ears perked up, for the notion that fractions with larger denominators are greater than those with smaller denominators (e.g., 1/8 is greater than 1/3) is common among elementary students. I heard in Lucy's assertion a space into which we could move more deliberately, a place in which the class could begin to extend their explicit, conceptual understanding of fractions. (Indeed, about two weeks later, two students brought forward the idea that "With fractions, the bigger the number on the bottom, the smaller the piece," a conjecture that another student quickly and spontaneously illustrated with models of 1/2, 1/3, 1/5, 1/7, 1/9 and 1/15.)

It was near the end of class. Sean raised his hand. "I think that to draw some number of pieces——like four——" and he got up from his seat and went to the board. He drew a rectangle. Turning to the class, he continued. "To draw four pieces, you just draw one less line——three." And he drew three lines inside the rectangle. "Because if you drew four lines"——and he drew one more line——"you would have *five* pieces, not four." I asked what others thought about Sean's conjecture. Several people said that they agreed with Sean, that they had found the same thing when they were making their drawings.

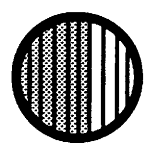
Betsy said she agreed, too. "And I have a different way to show it," she said. She picked up a pair of scissors and a piece of scrap paper. "This is a rectangle," she said. "If I make just three cuts, I will have four pieces." She cut the pieces, carefully, and then stuck them against the chalkboard with magnets.

People seemed intrigued, and some found her argument funny. But not everyone was convinced that this would *always* work. Whether or not something was always true was a question they had learned to ask when considering a mathematical generalization. Consideration of Sean's conjecture continued across several days, although many children began using it as they constructed their drawings. When I asked Daniel to explain why his drawing represented fourths, he explained that he had drawn a rectangle and put three lines in it. Riba argued, one day, that Sean's conjecture would always work because one line (or cut) always gave you "an edge"——that is, the other side of the region you are cutting. A few more were convinced by this logic. I was pleased that the children's use of area models for fractions had, among other things, generated opportunities for pattern finding and conjecturing such as this.

But difficult pedagogical questions about developing and structuring the use of representations continued to pop up. A few days later, I was standing by Maria's desk, watching her work on the problem of the day. I saw that she, struggling with the English words involved, had made a series of pictures of different fractions: 2/3, 4/5, 6/10, 3/11). She had drawn vertical lines inside circles:



$\frac{4}{5}$



$\frac{6}{10}$

I had noticed that other children had been making similar pictures, in spite of my attempt to push them toward rectangular models when we discussed Cassandra's solution to the cookie problem. I thought hard about what to do with Maria and the others. I could see that the students were genuinely excited by these new numbers. The pictures were helping them in figuring out one sense of what the numbers *meant*——that is, according to their working definition, that the bottom number told you how many parts and the top number how many "to take away." Contemplating their working definition helped to focus my deliberations.

The teacher is constantly in the position of having to listen to what her students are thinking and understanding and, at the same time, keep her eye on the mathematical horizon. Looking to that horizon, I could see that both the pictures and the definition of fractions were limited and problematic. These pictures did *not* divide equal parts. The numerator does not always indicate how many parts to "take away" from a whole. But, I realized, the children who were dividing rectangles were also, of course, not dividing into equal parts. They said things like, "Pretend it's equal." Such agreements were critical, otherwise drawings would have been entirely impossible.

Were——or should——the circles be regarded differently? After all, dividing circles so that the pieces are equal is much more complicated than doing so for rectangles. Yet, as a mathematical community, the students do need to agree on assumptions and shortcuts of language that facilitate communication. The students' explanation of fraction symbols——and what it suggested about their understanding of fractions——was also heading them for trouble, soon, in dealing improper fractions. As one student wailed, in trying to deal with 8/4, "if you take something and divide it into four parts, you can't take eight of them!"

When agreements within the discourse unknowingly (to the students) entail mathematical confusions or misconceptions, the teacher must be able to recognize them and to deliberate about the trade-offs. I decided, for the moment, to let the issue of circles pass and, instead, to urge directly the use of rectangles, saying that rectangles were "easier to use." With respect to the students' working definition of fractions, I decided to present the students with improper fractions. I began with the simple problem, "Which is more——2/4 or 4/2?" confronting them with a question that I thought would provoke a revision of their working understanding of fractions. I chose 4/2 for the provoking example of an improper fraction because I suspected that their robust intuitive understanding of halves would provide a semantic key for some students. Thinking about 4/2 as "four *halves*" was likely to make sense and convincingly dislodge the impossible alternative——"divide something into 2 pieces and take 4 of them." Using thirds or fourths might not offer this same wedge for their thinking.

We wended our way from the initial division-of-cookies problem into a serious exploration of fractions——as parts of wholes——including discrete sets, as numbers on the number line, and as operators. My decisions about representation——which to introduce, and how to structure their use, as well as how to respond to and shape the representations that the children brought——remained at the heart of my deliberations about the work.

Preparing Teachers to Construct Representational Contexts for Teaching Mathematics

Situations that mathematics teachers face——such as Cassandra's solution, Maria's circle pictures, or the group's working definition of fractions——highlight the complexity of constructing and using fruitful representational contexts for helping students develop understandings of mathematics. The examples in this paper spotlight the necessity for teachers to be able to hear and see mathematically what students are thinking. Teachers need to have multiple lenses and tools with which to deliberate about courses of action. They need to recognize, for example, that although equally spaced vertical lines inside a rectangle yield equal-sized pieces, inside a circle they do not. Teachers need to appreciate the value of Maria's explorations through the drawing of different fractions and to think about what might be gained——and what lost——if she were to work with more structured materials (e.g., fraction bars).

Teachers need to be able to hear the fallacies embedded in a definition of fractions that states that the top number is the amount you "take away" and be able to deliberate about what to do to help learners expand and deepen their understandings. Still, no answers, no certainties await us in deliberating about fruitful representations or their uses (cf., Ball, 1988; Floden and Clark, 1988). In helping students learn to understand and reason with fractions, justifiable decisions about representations——their construction, use, and adaptation——must be the product of a process of reasoning that can interweave deep understanding of fractions, and geometry, and measurement with ideas about mathematical reasoning and notions about nine-year-olds——what they understand and how they learn, what hooks them, what they might find exciting or interesting.

Current evidence about prospective and experienced teachers' understandings, assumptions, and ways of thinking about representation suggests that many do not focus on these sorts of considerations. Even as we become more sensitive in our understandings of the range of teaching that constitutes good practice, and of the accompanying inherent uncertainties and dilemmas (Floden and Clark, 1988; Lampert, 1985), we will need to attend with increasing care to what it will take to help people who have been steeped in traditional practice and conventional views of knowledge (Cohen, 1988) learn to teach mathematics for understanding.

What do we know about teachers of mathematics? There has been a recent growth in attention to and research on what prospective and experienced teachers know and believe——about mathematics, learners, learning, and teaching (e.g., Ball, 1988, 1990b, in press; Borko, Brown, Underhill, Eisenhart, Jones, and Agard, 1990; Carpenter, Fennema, Peterson, and Carey, 1988; Leinhardt and Smith, 1985; Martin and Harel, 1989; Peterson, Fennema, and Carpenter, in press; Peterson, Fennema, Carpenter, and Loef, 1989; Schram, Feiman-Nemser, and Ball, 1989; Schram, Wilcox, Lanier, and Lappan, 1988; Simon, 1990; Thompson, 1984; Tirosh and Graeber, 1990). In addition to providing insights into what they know and believe, these studies also begin to help us understand prospective and experienced teachers' representations and ways of reasoning. It is out of the interweaving of what they know and care about that their selection and use of representation is spun. What do they notice, consider, take into account? What decisions do they make about representation? These questions offer yet another critical perspective on the question of what teachers bring with them to teacher education related to the teaching of mathematics (Ball, 1988).

Two findings emerge consistently from these studies of teachers' knowledge and patterns of reasoning. One is that making mathematics fun and engaging is the central concern for many beginning and experienced teachers. Assuming that mathematics is not interesting to most students, they think that their role is to find ways to correct for that. In their study of eight prospective middle school teachers, for example, Borko et al. (1990) found that making mathematics class fun was central to these teachers' pedagogical reasoning. These researchers report uncovering a "pervasive belief" among the prospective teachers they studied that mathematics is inherently boring and hard to learn. In search of games that would lighten the load for students, the prospective teachers justified their choices most often in terms of how they would motivate or engage students rather than based on concerns for the mathematical content.

The prospective teachers whom we have interviewed (Ball, 1988; NCRTE, 1988) have also tended to be most concerned either with engaging students' interests or with being direct and clear about the specific mathematical content. In these studies, we interviewed elementary and secondary teacher education students on five different university campuses. The interviews were complemented with questionnaire data on a larger sample that included the sample of students who were interviewed. Like Borko et al. (1990), we found that many of the prospective teachers relied heavily, if not exclusively, on concerns for student interest: What will students find fun or interesting? What will they be able to relate to? The prospective teachers' focus on the learner was threaded with the assumption that if children are having fun or are able to "relate" to the material, they will learn. Making the contexts for learning mathematics fun was a top priority for many, rather than the links between the mathematics and students' thinking.

A second finding is that teachers' own mathematical experiences and understandings have not emphasized meaning and concepts. Although many teachers express commitments to focusing on concepts and emphasizing reasoning, a sizable proportion find that their own understanding of mathematics limits their ability to do so. Steeped in mathematics classes that stressed memorization and rules, these teachers face the need to revisit and revise the ways in which they learned mathematical ideas and procedures. For example, Borko et al. (1990) report that the prospective middle school mathematics teachers they followed talked consistently about the importance of concepts and meaning. Yet, even after their math methods course, they had trouble explaining why certain procedures, such as division of fractions or multiplication of decimals, work. They stumbled in trying to model mathematical concepts and procedures with concrete materials, pictures, or stories. When they used concrete models or pictorial representations in their teaching, they tended to use such representations rather perfunctorily and primarily as a means to keep and maintain students' attention and interest.

Like the prospective teachers interviewed by Borko and Brown (1990) and their colleagues, our prospective teachers' representations were also influenced by their own understandings of mathematics (e.g., of fractions, division, place value, area). Many of them were unable to unpack the conceptual underpinnings of the content, even when they completed teacher education. They also tended to continue to conceive mathematics as a body of rules. For example, at the conclusion of their studies, 69% of the elementary teacher candidates ($n = 83$) across our five sites were unable to select an appropriate representation for a division of fractions expression (e.g., 2-1/4 ÷ 1/2) from among four alternatives. And only 55% of the 22 secondary teacher education students——mathematics majors or minors——were able to select an appropriate representation at the end of their program.

Although these prospective teachers' responses revealed that they, too, had come to value manipulatives, and pictures, and diagrams, they were often unable to make use of these materials because of the thinness of their own mathematical knowledge. When asked what made representing division of fractions difficult, these teacher education students commented that it was hard (or impossible) to relate it to real life because, as one said, "you don't think in fractions, you think more in whole numbers." Another remarked, "I can't think of anything in the real world where you can divide by a fraction." Their stumblings were painful at times as they struggled to make sense using a mathematical background that had been "directed," as one student said, at getting the right answer, not at understanding *why*. Several commented that they didn't "like" fractions.

Studies of experienced teachers show that, as with prospective teachers, their assumptions about learners and their understandings of mathematics also shape the representational contexts they create (e.g., Ball, in press; Heaton, 1990; Leinhardt and Smith, 1985; Peterson, Fennema, Carpenter, and Loef, 1989; Schram, Feiman-Nemser, and Ball, 1989; Thompson, 1984), although "fun" is not always the dominant criterion. Like prospective teachers, many experienced teachers laud the use of manipulatives (Cohen, in press; Peterson, Fennema, Carpenter, and Loef, 1989; Schram, Feiman-Nemser, and Ball, 1989). Often they justify the value of manipulatives by explaining that when students see concepts concretely, they will remember them better (e.g., Cohen, in press; Schram, Feiman-Nemser, and Ball, 1989). Heaton's (1990) case study of Sandra Better spotlights an experienced fifth-grade teacher who eagerly gathered and used innovative activities. Her purposes, however, were focused primarily on motivating her students, especially girls; the mathematics for which the activities were designed tended to be distorted in the process.

Upper elementary grade teachers do, in general, seem less inclined to use concrete or visual representations than are primary teachers (Ball, in press; Remillard, 1990; Wiemers, in press; Wilson, in press). Experienced elementary teachers' orientations to and understandings of mathematics are also influential on the ways in which they represent mathematics. Leinhardt and Smith (1985) report that, although the teachers they interviewed could produce algorithms, they often did not understand the underlying mathematical concepts and relationships. This is not surprising when one considers that these rules were what was emphasized when they went to school. Teachers whose own understandings of the mathematics they teach is grounded in rules and algorithms tend to focus on mnemonics and other devices to help pupils remember the steps, rather than to create contexts for unpacking meanings (Ball, in press; Leinhardt and Smith, 1985; Remillard, 1990; Wilson, in press).

Teachers already have orientations to their role, to the nature and substance of mathematics, to what helps students learn. They already have patterns of reasoning and concerns that drive the kinds of decisions and compromises they make as they teach mathematics. These patterns are often quite different from what might be entailed in trying to interweave consideration of students' thinking with close analysis of the content to create productive representational contexts that can help students to develop mathematical understandings. For instance, a focus on making mathematics fun will justify some representations that are not grounded in meaning, that offer little opportunity for exploration or connections. Similarly, an orientation to and understanding of mathematics as rules and algorithms does not support a search for or use of conceptually grounded representational contexts.

Analyses of teaching——such as the analysis explored in this paper of the pedagogical reasoning underlying the construction of representational contexts——can help teacher educators and teachers consider the terrain of practice. Yet such analyses as this one are also insufficient. Changing one's practice is not a matter of merely acquiring information and techniques. Teachers who currently focus on devices that are catchy and that help students remember steps and rules cannot learn to construct the kinds of representational contexts explored in this paper simply by deciding to do so. Neither, even worse, can they construct such contexts merely by being exhorted to do so. The task is complex and uncertain. And taking teachers seriously as learners, considering where they are——what they already know and believe and how they reason, in relation to both content and students——together with what they are trying to do is key for those who would recommend changes in the practice of elementary mathematics teaching. Moreover, we need to continue to explore what kinds of experiences, supports, and structures can help teachers develop and change their practice.

Conclusion

Helping to develop new practices of mathematics teaching is no mean feat. Research can contribute to our work in this area; five lines of inquiry seem especially important. First, we need more theoretical and empirical research on representations in teaching particular mathematical content. For a given domain or topic, we need to construct and study an array of such representations and the contexts that might be structured for their use in classrooms. We need to map out conceptually and study empirically what students might learn from their interactions with them.

Second, we need to understand more about the processes of pedagogical deliberation in teaching mathematics for understanding. What kinds of dilemmas and issues arise——within particular mathematical content areas as well as more generally? Understanding what is entailed in trying to weave together concerns for mathematics with concerns for learners can contribute to helping people learn to teach.

Third, we need to understand better the role of mathematical understanding in teachers' pedagogical reasoning. What kinds or qualities of mathematical knowledge influence teachers' capacity to hear and interpret students' ideas and thinking? What kinds and qualities of mathematical knowledge support teachers' capacity to construct and use fruitful representational contexts?

Similarly, we need to learn more about the kinds and qualities of knowledge about learners and learning that contribute to teachers' ability to teach mathematics for understanding. What kinds or qualities of understandings, what dispositions and skills, influence teachers' capacity to hear and interpret students' ideas and thinking? What do teachers need to understand and be sensitive to in constructing and orchestrating helpful representations?

Final is the learning-to-teach question. What are alternative ways of helping people, whose entire experience with mathematics has been rule bound, often discouraging, and unsuccessful, learn to feel differently about themselves and to develop the dispositions, skills, and knowledge necessary to construct and use fruitful representational contexts in ways that go beyond making math class fun? How can they learn to transcend their own experiences with mathematics to consider other learners' experiences of and with mathematics?

I close this paper by returning the reader to the teacher's seat. The following coda returns to and pulls up the paper's central themes——of the interwoven threads of listening mathematically to children, sometimes following and sometimes gently pressing them onward, and of the issues entailed in figuring out, constructing, and using representational contexts in that process.

Coda[9]

Betsy : (working with Jeannie) How can we have this? (points to 4/2, written on the board)

Jeannie : I don't know.

Betsy : Four *twoths*?

Jeannie : We take something and divide it into two parts . . . and take *four* of those parts?

Betsy : I'm confused.

Jeannie : Me too.

Sheena : (walks up) Four *halves*, isn't it?

Betsy : Yeah, four *halves*! Halves are two parts. So . . .

Jeannie : So we need two cookies and cut them each in half, then we have four halves.

One, two, three, four. Twoths, I mean halves.

Overhearing this conversation, I realized the distance these girls had come. Beginning with an intuitive, inexplicit, and visual notion of one-half that they could draw, use and write, I had helped them travel into a new domain of numbers. Suddenly, looking back, the familiar looked, for a moment, strange.[10] One-*twoth*? But their comprehension of fractions had evolved into principled understanding of part-whole relationships and the symbolic notation for fractional quantities. And, consequently, a "2" in the denominator was no longer taken for granted: It had taken on explicit meaning. Ahead of these students still lie many excursions in the domain of rational numbers——into different interpretations and applications of rational numbers, as well as arithmetic with the rationals. They are launched now, with tools and ways of thinking that have built on and challenged the informal understandings they held.

[9] This is taken from my classroom after about two-and-a-half weeks of working formally on fractions.

[10] I would like to thank Janine Remillard for remarking how the fact that 1/2 suddenly looked like "one-twoth" is not unlike the ways in which young children overgeneralize as they extend their understandings in learning language. For example, a child

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may correctly say "I went"——until she discovers the "-ed" conjugation for the regular past tense. Then she is likely to go through a phase of saying "I goed." Similarly, my daughter, when she was four, was suddenly unable to write "45" correctly, although she had been able to do so for several months. Instead, I saw her, pausing, and then write "405"——an outgrowth of her new understanding of place value that had replaced an earlier, routine, recognition of two-digit numerals.

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